

À la Carte of Correlation Models: Which One to Choose?

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Abstract. In this paper we propose a copula contagion mixture model for correlated default times. The model includes the well known factor, copula, and contagion models as its special cases. The key advantage of such a model is that we can study the interaction of different models and their pricing impact. Specifically, we model the marginal default times to follow some contagion intensity processes coupled with copula dependence structure. We apply the total hazard construction method to generate ordered default times and numerically compare the pricing impact of different models on basket CDSs and CDOs in the presence of exponential decay and counterparty risk.

Keywords. copula contagion mixture model, exponential decay, counterparty risk, basket CDS and CDO.

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1 Introduction

The recent financial crisis has profound impact on the financial systems in the US, UK, and other major markets. Some giant banks and insurance companies either collapsed or had to be bailed out by the national governments. The excessive risk exposure of many banks to collateral debt obligations (CDOs) and credit default swaps (CDSs) has played the key role in this financial crisis. One may list many causes which have attributed to and aggravated the crisis, however, in this paper we only focus on the impact of correlation modelling on the pricing of these portfolio credit derivatives.

CDOs and CDSs had phenomenal growth in recent years until this financial crisis. The key in pricing and hedging these portfolio credit derivatives is the correlation modelling. There are mainly three approaches in the literature: conditional independence, copula, and contagion. Factor models are most popular due to their semi-analytic tractability. Many effective algorithms have been developed to characterize the portfolio loss distribution, see Andersen et al. (2003), Hull and White (2004) for recursive exact methods, and Glasserman (2004), Zheng (2006) for analytic approximation methods. Factor models may underestimate the portfolio tail risk and economic capital, see Das et al. (2007). Copula models are also popular, especially the Gaussian copula which is used in CreditMetrics, see Li (1999). Some copulas (Archimedean and exponential) are good to model extreme tail events and simultaneous defaults. There are some active recent debates on the usefulness of copulas in financial modelling and risk management, see Mikosch (2006) and many discussion papers on the same issue. Contagion models study the direct interaction of names in which the default intensity

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of one name may change upon defaults of other names and “infectious defaults” may develop, see Davis and Lo (2001), Jarrow and Yu (2001), Yu (2007). It is in general difficult to characterize the joint distribution of default times due to the looping dependence structure. For homogeneous portfolios there is a closed form formula for the density function of ordered default times, see Zheng and Jiang (2009). Monte Carlo method is often used to price CDOs and basket CDSs no matter which correlation model is used and provides benchmark results to test efficiency and accuracy of analytic and numerical algorithms.

It is interesting to know which model one should choose in pricing portfolio credit derivatives. We know different models give different values. If one uses the Gaussian copula, the swap rate for senior tranche of CDO is low due to the thin tail distribution of portfolio loss, on the other hand, if one uses the contagion model, the swap rate for the same senior tranche is much higher. However, one cannot simply say the contagion model is preferable to the Gaussian copula because it provides higher swap rates for senior tranche. It all depends on the underlying model assumptions. These correlation models are defined under different frameworks and are difficult to compare directly their pricing impact. It is therefore beneficial to have a unified model which covers all three known models as special cases. One may then extract the information of the interaction of these models and may give a more balanced view on which model one should choose for a specific application.

In this paper we suggest a general copula contagion mixture model which includes factor, copula, and contagion models as its special cases. The key advantage of such a model is that we can study the interaction of different models and their pricing impact. Specifically, we model the marginal default times to follow some exponential decay contagion intensity processes coupled with some copula dependence structure. This is not a Markov process model and cannot be solved with the standard Kolmogorov equations or matrix exponentials, see Frey and Backhaus (2004), Herbertsson and Rootzen (2006). Although there are analytic pricing formulas for some special cases, we choose to use the Monte Carlo method to price CDOs and basket CDSs, which is reliable, accurate and efficient with some optimized numerical procedure.

The paper is organized as follows: section 2 describes the copula contagion mixture model and the relation with the known models, section 3 applies the model to price CDOs and basket CDSs and discuss the impact of interaction of different models, section 4 concludes.

2 The Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space, where P is the martingale measure and $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration satisfying the usual conditions. Let τ_i be the default time of name i , $N_i(t) = 1_{\{\tau_i \leq t\}}$ the default indicator process of name i , $\mathcal{F}_t^i = \sigma(N_i(s) : s \leq t)$ the filtration generated by default process N_i , $i = 1, \dots, n$, and $\mathcal{F}_t = \mathcal{F}_t^1 \vee \dots \vee \mathcal{F}_t^n$ the smallest σ -algebra needed to support τ_1, \dots, τ_n . Assume that τ_i possesses a nonnegative \mathcal{F}_t predictable intensity process $\lambda_i(t)$ satisfying $\mathbb{E}[\int_0^t \lambda(s) ds] < \infty$ for all t , and the compensated process

$$M_i(t) = N_i(t) - \int_0^{t \wedge \tau_i} \lambda_i(s) ds$$

is an \mathcal{F}_t martingale. Given $\tau_j = t_j$, $j \in J_k = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$, satisfying $0 = t_{j_0} < t_{j_1} < \dots < t_{j_k}$ and $\tau_i > t > t_{j_k}$ for $i \notin J_k$, the conditional hazard rate of τ_i at time t is given

by

$$\lambda_i(t|t_{J_k}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(t < \tau_i \leq t + \Delta t | \tau_j = t_j, j \in J_k)$$

where t_{J_k} is a short form for $(t_{j_1}, \dots, t_{j_k})$.

In copula modelling of default times it is normally assumed that intensity processes are independent of default times of other names, i.e., $\lambda_i(t|t_{J_k}) = \lambda_i(t)$ for all t . The marginal distribution functions of default times τ_i are given by $F_i(t) = P(\tau_i \leq t) = \mathbb{E}[1 - \exp(-\int_0^t \lambda_i(s) ds)]$ (if λ_i are stochastic processes) and standard uniform variables $F_i(\tau_i)$, $i = 1, \dots, n$, have a joint distribution function C , a given copula. It is easy to generate default times τ_i with the Monte Carlo method. One can simply first generate standard uniform variables U_i with copula C and sample paths of λ_i and then find the default times τ_i by

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(s) ds \geq E_i \right\}$$

where $E_i = \ln(1 - U_i)$, $i = 1, \dots, n$, are correlated standard exponential variables. In particular, if $\lambda_i(t) = a_i$, then $\tau_i = E_i/a_i$.

There has been extensive research in literature on factor modelling of default times. These models are all special cases of copula modelling of default times. For example, the well-known Gaussian factor model is given by

$$X_i = \rho Z + \sqrt{1 - \rho^2} Z_i$$

where Z, Z_1, \dots, Z_n are independent standard normal variables and ρ is a constant satisfying $|\rho| \leq 1$. Z is often interpreted as a systematic factor and Z_i idiosyncratic factors. If we set $U_i = \Phi(X_i)$, where Φ is the standard normal distribution function, then the distribution of U_i is a Gaussian copula given by

$$C(u_1, \dots, u_n) = \Phi_{m, \Gamma}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$$

where $\Phi_{m, \Gamma}$ is the n -variate standard normal distribution function with mean vector $m = 0$ and correlation matrix Γ that has diagonal elements 1 and all other elements ρ^2 . Factor models are appealing from model interpretation and conditional independence point of view. The corresponding copulas may have some complex forms, but in general it is easy to generate correlated standard uniform variables due to the special structure of factor models. From mathematics point of view there is no need to treat them separately if we know how to generate standard uniform variables U_i from given copulas C .

In contagion modelling of default times the intensity processes $\lambda_i(t|t_{J_k})$ depend on default times of other names and standard uniform variables $F_i(\tau_i)$ are assumed to be independent. The marginal distribution functions F_i of default times τ_i cannot be simply expressed in terms of $\lambda_i(s)$ as information of other default times is needed to characterize the whole intensity process paths and there is a “looping” phenomenon. Although it is difficult to characterize the marginal and joint distributions of default times it is easy and straightforward to generate ordered default times τ^i with the total hazard construction method. One can first generate independent standard uniform variables U_i and set $E_i = -\ln(1 - U_i)$, $i = 1, \dots, n$, then find default times one by one as follows: To find the first default time τ^1 and the corresponding name j_1 , set

$$j_1 = \operatorname{argmin}_{j=1, \dots, n} \left\{ t_j > 0 : \int_0^{t_j} \lambda_j(s) ds \geq E_j \right\}$$

and $\tau^1 = t_{j_1}$ and $J_1 = \{j_1\}$, where $\lambda_j(s)$ are unconditional hazard rates of names j at time s . To find the k th default time τ^k and the corresponding name j_k for $k \geq 2$, set

$$j_k = \operatorname{argmin}_{j \notin J_{k-1}} \left\{ t_j > \tau_{j_{k-1}} : \int_0^{t_j} \lambda_j(s | t_{J_{k-1}}) ds \geq E_j \right\}$$

and $\tau^k = t_{j_k}$ and $J_k = J_{k-1} \cup \{j_k\}$.

We suggest a copula contagion mixture model which covers both copula model and contagion model as special cases. Specifically, we assume that the intensity processes λ_i may depend on default times of other names and standard uniform variables $F_i(\tau_i)$ have a joint distribution C . This is a natural generalization of pure copula models and pure contagion models. One can easily generate default times with the total hazard construction method. The only difference with the pure contagion model is that one generates standard uniform variables U_i from a given copula C , not necessarily from the product copula which corresponds to the pure contagion model. The key advantage of this new mixture model is that, instead of studying three well known models in isolation, we can explore their interaction and their joint pricing impact on CDOs and basket CDSs.

We now impose some structure to the intensity processes. To simplify the notation and highlight the key point, we assume a homogeneous portfolio. The discussion is the same for general heterogeneous intensity processes except the expression is more complicated. We assume the intensity processes have the following structure

$$\lambda_i(t) = a \left(1 + \sum_{j=1, j \neq i}^n c e^{-d(t-\tau_j)} 1_{\{\tau_j \leq t\}} \right), \quad i = 1, \dots, n, \quad (1)$$

where a, c, d are positive constants. (These parameters can be deterministic functions of t or even some stochastic processes, the discussion is essentially the same, see Zheng and Jiang (2009).) a is the unconditional default intensity, c is the contagion rate, and d is the exponential decay rate. When $d = 0$ we may introduce the default state space and use the Markov Chain to study the joint distribution of default times. Apart from this extreme case the intensity processes (1) are non-Markov.

For homogeneous intensity processes (1) without exponential decay ($d = 0$) we can simplify the total hazard construction method. This is because we only need to know the number of defaults at time t but not the identities of names which have defaulted. We can generate τ^k as follows:

- Step 1.* Generate correlated standard uniform variables U_i , $i = 1, \dots, n$, from the copula C .
- Step 2.* Set $E_i = -\ln(1 - U_i)$, $i = 1, \dots, n$, and sort E_i in increasing order to get E_i^* with $E_1^* < E_2^* < \dots < E_n^*$.
- Step 3.* Find ordered default times τ^k by setting

$$\tau^1 = \frac{E_1^*}{a}, \quad \tau^k = \tau^{k-1} + \frac{E_k^* - E_{k-1}^*}{a(1 + (k-1)c)} \quad \text{for } k = 2, \dots, n. \quad (2)$$

The density function of the k th default time τ^k is given by (for $d = 0$)

$$f_{\tau^k}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} a e^{-\beta_j a t}$$

where $\beta_j = (n - j)(1 + jc)$ and $\alpha_{k,j}$ are constants depending on c, k, j, n and have explicit expressions. For example, $f_{\tau^1}(t) = nae^{-nat}$ which shows that the contagion has no influence on the first default time, and

$$f_{\tau^2}(t) = \begin{cases} \frac{n(n-1)(1+c)a}{(1+(1-n)c)}(-e^{-nat} + e^{-(n-1)(1+c)at}), & \text{if } c \neq 1/(n-1) \\ (na)^2 te^{-nat}, & \text{if } c = 1/(n-1) \end{cases}$$

which implies that the contagion affects the second and all subsequent default times. We can then derive the analytic pricing formulas for basket CDSs and CDOs, see Zheng and Jiang (2009).

For general homogeneous intensity processes (1) we cannot use (2) to generate ordered default times. The computation is slightly more involved. Steps 1 to 2 are the same and so is the first default time τ^1 . Assume ordered default times $\tau^1, \dots, \tau^{k-1}$ have already been generated for some $k \geq 2$. Now we want to generate τ^k . Let $\tau^{k-1} \leq t \leq \tau^k$. The total hazard accumulated by name k at time t is

$$\int_0^t \lambda_k(s) ds = \sum_{j=1}^{k-1} \int_{\tau^{j-1}}^{\tau^j} a \left(1 + \sum_{i=1}^{j-1} ce^{-d(s-\tau^i)} \right) ds + \int_{\tau^{k-1}}^t a \left(1 + \sum_{i=1}^{k-1} ce^{-d(s-\tau^i)} \right) ds$$

where $\tau^0 = 0$ and $\sum_{i=1}^0 = 0$ by convention. Simplifying the above expression we get

$$\int_0^t \lambda_k(s) ds = at + \frac{ac}{d} \sum_{i=1}^{k-1} \left(1 - e^{-d(t-\tau^i)} \right).$$

The τ^k is determined by the relation $\int_0^{\tau^k} \lambda_k(s) ds = E_k^*$. Define

$$F_k(t) := at + \frac{ac}{d} \sum_{i=1}^{k-1} \left(1 - e^{-d(t-\tau^i)} \right) - E_k^*.$$

Then τ^k is a root of nonlinear equation $F_k(t) = 0$. Since $F'_k(t) > 0$ and $F''_k(t) < 0$ function F_k is strictly increasing and strictly concave. Observe also that from $F_{k-1}(\tau^{k-1}) = 0$ we have

$$F_k(\tau^{k-1}) = a\tau^{k-1} + \frac{ac}{d} \sum_{i=1}^{k-1} \left(1 - e^{-d(\tau^{k-1}-\tau^i)} \right) - E_k^* = E_{k-1}^* - E_k^* < 0$$

and $F_k(\infty) = \infty$. There is a unique root of equation $F_k(t) = 0$ on the interval $[\tau^{k-1}, \infty)$. The special structure of function F_k guarantees that the Newton algorithm with an initial iterating point τ^{k-1} converges quadratically to the root τ^k . We can now summarize Sept 3 in the presence of exponential decay rate $d > 0$ as follows.

Sept 3. Set $\tau^1 = E_1^*/a$ and find the k th default time τ^k by solving numerically the equation $F_k(t) = 0$ with the Newton algorithm and the initial iterating point τ^{k-1} for $k = 2, \dots, n$.

We now discuss the impact of exponential decay rate d on ordered default times τ^k . From $F'_k(t) = a + ac \sum_{i=1}^{k-1} e^{-d(t-\tau^i)}$ we know that $F'_k(t)$ is a strictly decreasing function of d for $t > \tau^{k-1}$. If $d = 0$ we have $F'_k(t) = a + ac(k-1)$ and F_k is a linear function

$$F_k(t) = E_{k-1}^* - E_k^* + (a + ac(k-1))(t - \tau^{k-1}).$$

The k th default time τ^k is given by (2) as expected. If $d = \infty$ we have $F'_k(t) = a$ and F_k is again a linear function

$$F_k(t) = E_{k-1}^* - E_k^* + a(t - \tau^{k-1}).$$

The k th default time is given by $\tau^k = \tau^{k-1} + (E_k^* - E_{k-1}^*)/a$, or equivalently, $\tau^k = E_k^*/a$, which corresponds to the case when there is no contagion effect. For any other d the k th default time τ^k lies between these two extreme cases. We conclude that the smaller the exponential decay rate, the stronger the contagion effect and the sooner the ordered default times, which makes CDO and basket CDS riskier and demands higher spreads.

3 Numerical Tests

We can now value the basket CDS and CDO with the copula contagion mixture model. We assume homogeneous intensity processes (1) to simplify the computation, but the same method can be applied to general intensity processes. For both basket CDS and CDO we assume that T is the maturity of the contract, $t_1 < t_2 \dots < t_N$ are swap rate payment dates, $t_0 = 0$ is the initial time and $t_N = T$ is the terminal time, R is the recovery rate, r is the riskless interest rate, and $B(t) = e^{-rt}$ is the discount factor at time t .

To price basket CDS we assume S_k is the annualized k th default swap rate. The expected value of the contingent leg at time 0 is equal to

$$\mathbb{E}[(1 - R)B(\tau^k)1_{\{\tau^k \leq T\}}]$$

and that of the fee leg with accrued interest is equal to

$$S_k \mathbb{E} \left[\sum_{i=1}^N \left((t_i - t_{i-1})B(t_i)1_{\{\tau^k > t_i\}} + (\tau^k - t_{i-1})B(\tau^k)1_{\{t_{i-1} < \tau^k \leq t_i\}} \right) \right].$$

We can easily find the swap rate S_k with the Monte Carlo method by generating ordered default times τ^k .

To price CDO we assume k_l , $l = 0, \dots, M - 1$, are attachment points of tranches l with $0 = k_0 < k_1 < \dots < k_M = 1$, $\Delta k_l = k_l - k_{l-1}$ are tranche sizes for $l = 1, \dots, M$, the cumulative percentage portfolio loss at time t is given by

$$L(t) = \sum_{k=1}^n \frac{k}{n} 1_{\{\tau^k \leq t < \tau^{k+1}\}}$$

with $\tau^0 = 0$ and $\tau^{n+1} = \infty$, the cumulative tranche l loss at time t is given by

$$L_l(t) = (L(t) - k_{l-1})1_{\{k_{l-1} \leq L(t) \leq k_l\}} + \Delta k_l 1_{\{L(t) > k_l\}}.$$

Assume S_l is the swap rate of tranche l . The expected value of the contingent leg for tranche l loss at time 0 is given by (note $L_l(0) = 0$)

$$\mathbb{E} \left[\sum_{i=1}^N B(t_i) (L_l(t_i) - L_l(t_{i-1})) \right]$$

ρ/c	tranche	0.0	0.3	3.0
0.0	0-0.15	0.0740	0.0890	0.2360
	0.15-0.3	0.0000	0.0003	0.1052
	0.3-1	0.0000	0.0000	0.0199
0.5	0-0.15	0.0682	0.0843	0.1553
	0.15-0.3	0.0042	0.0164	0.1020
	0.3-1	0.0001	0.0022	0.0596
0.9	0-0.15	0.0326	0.0373	0.0488
	0.15-0.3	0.0147	0.0242	0.0439
	0.3-1	0.0044	0.0157	0.0405

Table 1: CDO rates with the Gaussian copula contagion mixture model.

and that of the fee leg for tranche l is

$$S_l \mathbb{E} \left[\sum_{i=1}^N (t_i - t_{i-1}) B(t_i) (\Delta k_l - L_l(t_i)) \right].$$

We can again easily find the swap rate S_l with the Monte Carlo method.

To generate ordered default times we must first generate correlated standard uniform variables U_i , $i = 1, \dots, n$. We use three different copulas to generate U_i . The first one is the product copula and U_i are simply independent standard uniform variables. The second one is the exponential copula and U_i are generated as follows: first generate $n + 1$ independent exponential variables T_0, T_1, \dots, T_n , where T_0 has parameter c_0 and T_1, \dots, T_n have parameter c_1 , then set $S_i = \min(T_0, T_i)$, and finally define $U_i = 1 - \exp(-(c_0 + c_i)S_i)$, $i = 1, \dots, n$. This is the simplest exponential copula which models simultaneous jumps as well as individual jumps, see Giesecke (2003) for more details on exponential copulas and Xu and Zheng (2009) for their applications in modelling portfolio asset price processes. The third model is the Gaussian copula and U_i are generated as follows: first generate $n + 1$ independent standard normal variables Z, Z_1, \dots, Z_n , then set $X_i = \rho Z + \sqrt{1 - \rho^2} Z_i$, $i = 1, \dots, n$, and finally define $U_i = \Phi(X_i)$, $i = 1, \dots, n$. This is the most popular model used in financial institutions for pricing portfolio derivatives.

We have used the following data in numerical tests: number of names $n = 40$, riskless interest rate $r = 0.05$, time to maturity $T = 3$, number of payments $N = 6$ with equally spaced time intervals, unconditional intensity rate $a = 0.01$, recovery rate $R = 0.5$, exponential decay rate $d = 0$, and number of simulations is 1 million.

Table 1 lists CDO rates computed with the Gaussian copula contagion mixture model with different c and ρ . We can see that swap rates increase if c increases, which is expected as higher c causes higher contagion and more defaults. $c = 0$ corresponds to the Gaussian factor model. As ρ increases swap rates for equity tranche decrease while those for mezzanine and senior tranches increase, a well known fact. $\rho = 0$ corresponds to the pure contagion model (or the product copula contagion mixture model) and we see c has huge impact on swap rates for mezzanine and senior tranches. When both c and ρ are positive, we see swap rates for senior tranche are greater than those with the pure contagion model ($\rho = 0$) and the pure factor model ($c = 0$). It is interesting to note that swap rates for mezzanine tranche

c	0			0.3			3		
k	ProdC	ExpC	GausC	ProdC	ExpC	GausC	ProdC	ExpC	GausC
1	0.2024	0.1575	0.1153	0.2024	0.1575	0.1153	0.2024	0.1575	0.1153
2	0.0634	0.0697	0.0508	0.0769	0.0811	0.0573	0.1401	0.1249	0.0855
5	0.0010	0.0026	0.0105	0.0052	0.0104	0.0197	0.0836	0.0866	0.0620
10	0.0000	0.0000	0.0014	0.0000	0.0001	0.0072	0.0486	0.0582	0.0492
20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0016	0.0163	0.0263	0.0369
30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0024	0.0061	0.0274
0–0.15	0.0740	0.0742	0.0682	0.0890	0.0923	0.0843	0.2360	0.2218	0.1553
0.15–0.3	0.0000	0.0000	0.0042	0.0003	0.0011	0.0164	0.1052	0.1246	0.1020
0.3–1	0.0000	0.0000	0.0001	0.0000	0.0000	0.0022	0.0199	0.0314	0.0596

Table 2: Comparison of basket CDS and CDO rates with the product copula contagion mixture model (ProdC), the exponential copula contagion mixture model (ExpC) with $c_0 = 0.01$ and $c_1 = 0.1$, and the Gaussian copula contagion mixture model (GausC) with $\rho = 0.5$.

decrease as ρ increases when $c = 3$, an opposite phenomenon to the case when $c = 0$. This is not surprising because when $c = 3$ the default intensity increases quickly for surviving names and many more names are likely to default, in other words, the mezzanine tranche behaves increasingly like the equity tranche, and therefore as ρ increases the corresponding swap rates actually decrease. For the same reason the senior tranche behaves increasingly like the mezzanine tranche and its swap rates increase and then decrease as ρ increases. We have also done numerical tests for $a = 0.1$ and found that all tranches behave like the equity tranche and swap rates decrease as ρ increases even when $c = 0.3$.

Table 1 may shed some light on the cause of recent financial crisis. Before the full scale credit crunch, the housing and stock markets were booming, the credit was cheaply and easily available, few individuals and companies defaulted, and default rates from rating agencies were very low. Portfolio credit derivatives such as synthetic CDOs were in high demand. The Gaussian factor model (corresponding to $c = 0$ in the table) was the most popular model used in financial institutions to price these securities. Table 1 shows that for the senior tranche (0.3-1) the risk is almost negligible for $\rho = 0.5$, and is still very small even for unlikely $\rho = 0.9$. It seemed that the underwriter of CDS for CDO senior tranches could make huge profit from premium fees with little risk, almost like “free lunch with vanishing risk”. However, when there is contagion, which is the case for synthetic CDOs (the actual loss can be many times over the nominal loss), the risk for the senior tranche is much higher even in good economy ($a = 0.01$). When $c = 3$ and $\rho = 0.5$, the swap rate for the senior tranche is 0.0596, in sharp contrast to 0.0001 when $c = 0$ and $\rho = 0.5$. Mis-pricing of synthetic CDOs could be one of the causes which led to the financial crisis of underwriters of CDS for these synthetic portfolio credit derivatives.

Table 2 lists basket CDS and CDO rates computed with three copula contagion mixture models. The copulas used are the product copula, the exponential copula with $c_0 = 0.01$ and $c_1 = 0.1$ (individual jumps is much more likely than a systematic jump), and the Gaussian copula with $\rho = 0.5$. The results for basket CDS are mixed with no single model dominating the others in pricing. Contagion has no influence to the 1st default CDS rate and the product

k	$d = 0$	$d = 1$	$d = 10$	$d = 100$	$d = \infty$
1	0.1153	0.1153	0.1153	0.1153	0.1153
2	0.0855	0.0761	0.0564	0.0514	0.0508
5	0.0620	0.0482	0.0175	0.0111	0.0105
10	0.0492	0.0348	0.0053	0.0017	0.0014
20	0.0369	0.0230	0.0008	0.0001	0.0000
30	0.0274	0.0137	0.0001	0.0000	0.0000
0–0.15	0.1553	0.1323	0.0810	0.0696	0.0682
0.15–0.3	0.1020	0.0727	0.0127	0.0048	0.0042
0.3–1	0.0596	0.0328	0.0012	0.0002	0.0001

Table 3: Comparison of basket CDS and CDO rates with exponential decay Gaussian copula contagion mixture model, data used are $a = 0.01$, $\rho = 0.5$, and $c = 3$.

copula produces the highest rate. When there is no contagion ($c = 0$) or low contagion ($c = 0.3$) the Gaussian copula dominates the swap rates for all k but the first few. When there is high contagion ($c = 3$) the results are more mixed with the Gaussian copula dominating for large k and the other copulas for small k . The results for CDO are also mixed. For equity tranche the product and exponential copulas produce similar rates which are higher than those from the Gaussian copula. For mezzanine and senior tranches the Gaussian copula gives much higher rates than the other two copulas do except when contagion is high ($c = 3$) and the rates from the other two copulas are also increased significantly. The difference between swap rates using different copula contagion mixture models is substantial.

Table 3 lists basket CDS and CDO rates with the exponential decay Gaussian copula contagion mixture model. The data used are $a = 0.01$, $c = 3$, $\rho = 0.5$, and different decay rates. $d = 0$ corresponds to the Gaussian copula contagion mixture model without decay and $d = \infty$ to the case without contagion effect. It is clear that as d increases, basket CDS and CDO rates decrease. The exponential decay has much greater impact to the k th default rates for larger k than for smaller k . The same phenomenon is observed for CDO rates, that is, the exponential decay has much greater impact to senior tranche rates than to junior tranche rates. Basket CDS and CDO rates are highly sensitive to exponential decay rates d , which requires an accurate estimation of d in calibration if one is to use it in pricing.

We have done some numerical tests on pricing of CDOs and basket CDSs in the presence of counterparty risk. Assume the intensity processes (1) for underlying names in the portfolio and assume the intensity process of default time τ^B of the counterparty is given by

$$\lambda_B(t) = a_B \left(1 + \sum_{i=1}^n c_B 1_{\{\tau^i \leq t\}} \right)$$

where a_B is the unconditional default intensity and c_B the contagion rate. Note that the hazard rate process λ_B of the counterparty is influenced by defaults of names in the portfolio, but not vice versa. This follows the observation in Leung and Kwok (2005) and Yu (2007) that the contagion of the counterparty on underlying names does not affect CDS pricing.

To price basket CDS we only need to compute the expected value of the contingent leg

	c	0.0		0.3		3.0	
ρ	tranche	GausC	GausCCR	GausC	GausCCR	GausC	GausCCR
0.0	0-0.15	0.0740	0.0740	0.0890	0.0889	0.2360	0.2347
	0.15-0.3	0.0000	0.0000	0.0003	0.0003	0.1052	0.1027
	0.3-1	0.0000	0.0000	0.0000	0.0000	0.0199	0.0188
0.5	0-0.15	0.0682	0.0680	0.0843	0.0841	0.1553	0.1521
	0.15-0.3	0.0042	0.0040	0.0164	0.0160	0.1020	0.0968
	0.3-1	0.0001	0.0001	0.0022	0.0020	0.0596	0.0500
0.9	0-0.15	0.0326	0.0326	0.0373	0.0364	0.0488	0.0421
	0.15-0.3	0.0147	0.0144	0.0242	0.0232	0.0439	0.0355
	0.3-1	0.0044	0.0040	0.0157	0.0137	0.0405	0.0291

Table 4: CDO rates with the Gaussian copula contagion mixture model (GausC) and the same model with counterparty risk (GausCCR).

and the fee leg at time 0, given respectively by

$$\mathbb{E} \left[(1 - R)B(\tau^k)1_{\{\tau^k \leq T, \tau^B \geq \tau^k\}} \right]$$

and

$$S_k \mathbb{E} \left[\sum_{i=1}^N \left((t_i - t_{i-1})B(t_i)1_{\{\tau^k > t_i, \tau^B > t_i\}} + (\tau^k - t_{i-1})B(\tau^k)1_{\{t_{i-1} < \tau^k \leq t_i, \tau^B > \tau^k\}} \right) \right].$$

Similarly to price CDO tranche l we only need to compute the expected value of the contingent leg and the fee leg at time 0, given respectively by

$$\mathbb{E} \left[\sum_{i=1}^N B(t_i)(L_l(t_i) - L_l(t_{i-1}))1_{\{\tau^B > t_i\}} \right]$$

and

$$S_l \mathbb{E} \left[\sum_{i=1}^N (t_i - t_{i-1})B(t_i)(\Delta k_l - L_l(t_i))1_{\{\tau^B > t_i\}} \right]$$

We can easily find swap rates with the total hazard construction method by generating ordered default times τ^k and counterparty default time τ^B .

Table 4 lists the swap rates of all tranches with and without counterparty risk. We have used the data $a_B = a/10 = 0.001$ and $c_B = c$. The counterparty is much less likely to default than those names in the portfolio. However, defaults of names in the portfolio increase the default intensity of the counterparty. It can be observed that the equity tranche is least affected by the counterparty risk while the senior tranche is most affected. This is expected as the counterparty is much more likely to default due to the contagion effect from defaults of names in equity and mezzanine tranches, and therefore the senior tranche investors require higher compensation for increased counterparty risk.

4 Conclusions

In this paper we have suggested a general exponential decay copula contagion mixture model which unifies the factor model, copula model, and contagion model. The key advantage is that one can investigate the interaction of these models and its pricing impact on basket CDS and CDO. The ordered default times can be easily generated with the total hazard construction method. We have done some numerical tests and compared basket CDS and CDO rates with three copula (product, exponential, and Gaussian) contagion mixture models and found that there is no model dominating the others in pricing although one model may affect much greatly the pricing of some parts of basket CDS and CDO than the other models do. The difference of swap rates computed with different models can be substantial. The exponential decay rate has great impact on senior tranche rates and k th default rates for large k . We have also compared pricing results when there is contagion counterparty risk. Our conclusion is that one has to be cautious in pricing basket CDS and CDO when a particular model is used as different models may greatly influence the portfolio loss distribution and can significantly affect the resulting swap rates. We should not put all blames on the “misplaced reliance on sophisticated maths” for the recent financial crisis, see Cookson (2009). No model is best for all purposes. Stress test and other risk control procedures should be in place to withstand the potential loss due to the wrong choice of models.

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